SEARCH IN AI

CONSTRAINT SATISFACTION PROBLEMS (CSP)

given a configuration
valid configuration
solve a set of constraints

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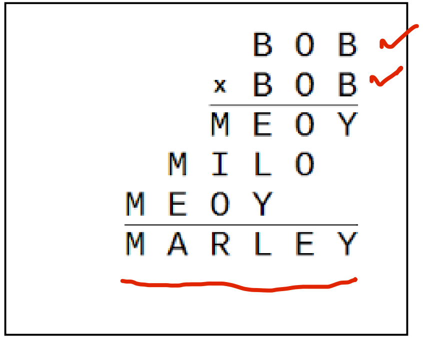
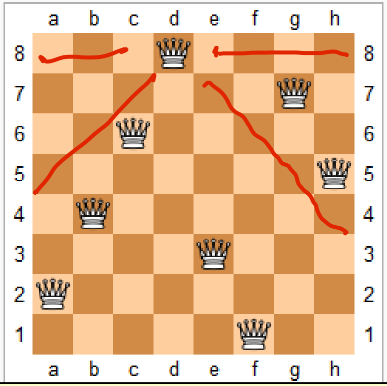
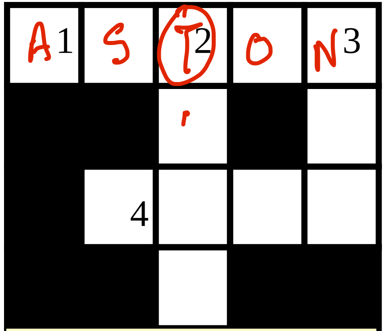
Constraint Satisfaction Problems (CSPs)



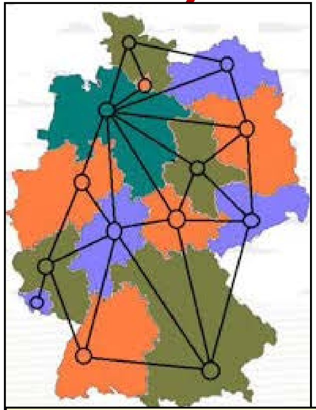
CRYPTARITHMETIC PUZZLE



N-QUEENs



CROSSWORD PUZZLE

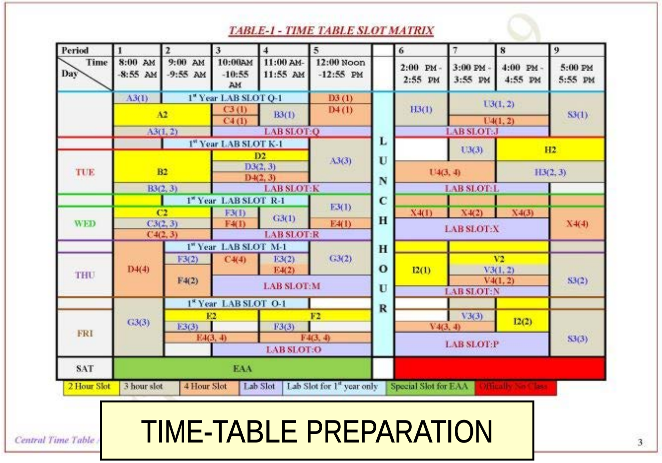


MAP COLOURING

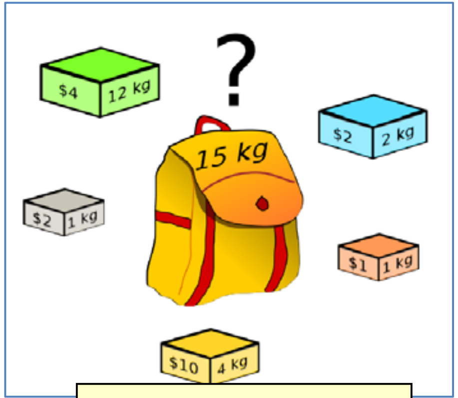
International Departures

Flight No	Destination	Time	Gate	Remarks
CX7183	Berlin	7:50	A-11	Gate closing
QF3474	London	7:50	A-12	Gate closing
BA372	Paris	7:55	B-10	Boarding
AY6554	New York	8:00	C-33	Boarding
KL3160	San Francisco	8:00	F-15	Boarding
BA8903	Manchester	8:05	B-12	Gate lounge open
BA710	Los Angeles	8:10	C-12	Check-in open
QF3371	Hong Kong	8:15	F-10	Check-in open
MA4866	Barcelona	8:15	F-12	Check-in at kiosks
CX7221	Copenhagen	8:20	G-32	Check-in at kiosks

AIRLINE GATE SCHEDULING



TIME-TABLE PREPARATION



KNAPSACK

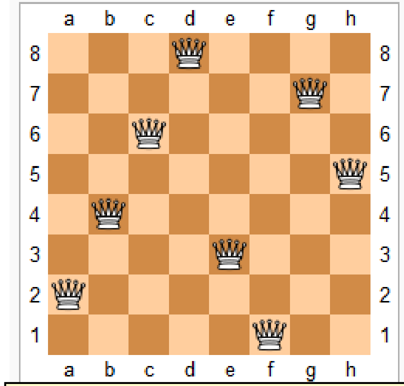
Basic CSP Formulation

- Variables
  - A Finite Set of Variables V_1, V_2, …., V_n
- Domains
  - Each Variable has a Domain D_1, D_2, …., D_n from which it can take a value.
  - The Domains may be discrete or continuous domains
- Satisfaction Constraints
  - A Finite Set of Satisfaction Constraints, C_1, C_2, …C_m
  - Constraints may be unary, binary or be among many variables of the domain
  - All Constraints have a Yes / No Answer for Satisfaction given values of variables
- Optimization Criteria (Optional)
  - A Set of Optimization Functions O_1, O_2, ….O_p
  - These Optimization Functions are typically max or min type
- Solution
  - To Find a Consistent Assignment of Domain Values to each Variable so that All Constraints are Satisfied and the Optimization Criteria (if any) are met.

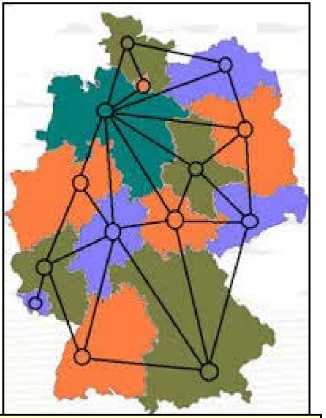
Formulating CSPs

```
        B O B
      x B O B
      M E O Y
    M I L O
  M E O Y
  M A R L E Y
```

CRYPTARITHMETIC PUZZLE



N-QUEENs



MAP COLOURING

1. VARIABLES
2. DOMAINS
3. SATISFACTION CONSTRAINTS
4. OPTIMIZATION CRITERIA
5. SOLUTION

$\underbrace{B, O, M, E, Y, I, L, R}_{\text{Variables}}$

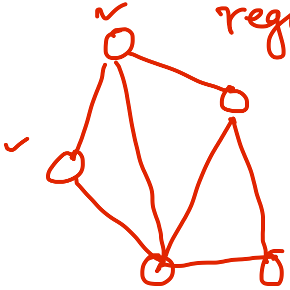
Domain : {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Constraints :- Uniqueness
Multiplication operator

row is a variable

D : a, b, ..., h

constraints



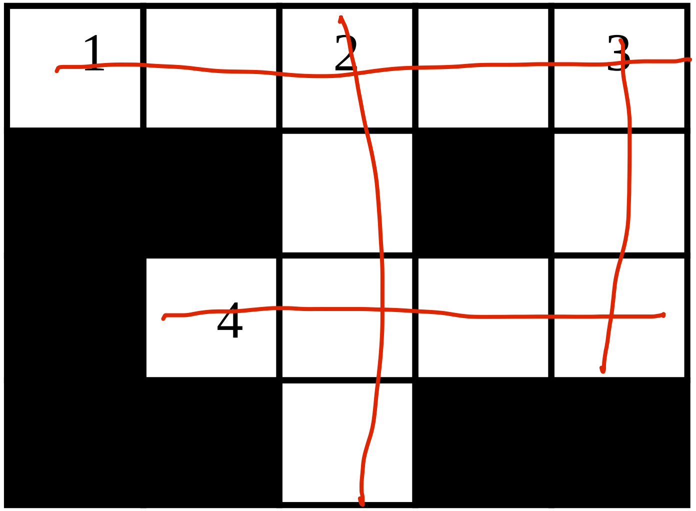
region : variable

{1, 2, 3}

Domain

min no. of colours

Formulating CSPs: Crossword



Word List:

astar, happy, hello, hoses, live, load, loom, peal, peel, save, talk, ant, oak, old

1. VARIABLES
2. DOMAINS
3. SATISFACTION CONSTRAINTS
4. OPTIMIZATION CRITERIA
5. SOLUTION

variables: 1, 2, 3, 4

Domain: Word list

constraints

(1, 2)
(1, 3)
(2, 4)
(3, 4)

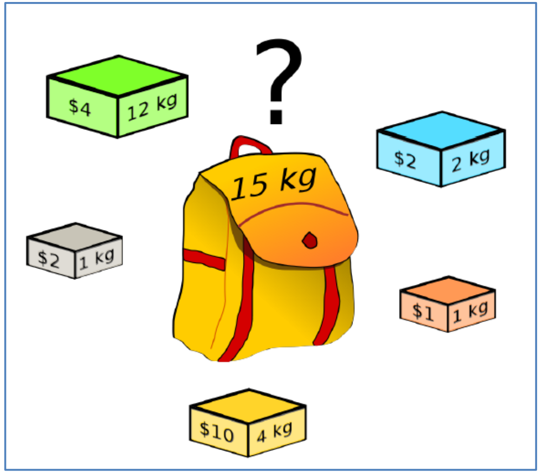
Formulating CSPs: Flight Gate Scheduling

Flight No	Dep Time	G Start	G End
F1	7:00	6:15	7:15
F2	8:30	7:45	8:45
F3	7:45	7:00	8:00
F4	9:45	9:00	10:00
F5	10:00	9:15	10:15
F6	9:00	8:15	9:15
F7	11:00	10:15	11:15

7 flights

1. VARIABLES → $F_1, F_2, \ldots, F_7$
2. DOMAINS → Gate Nos $\{G_1, G_2, G_3, G_4\}$
3. SATISFACTION CONSTRAINTS → Flights having overlapping times cannot be assigned the same gate
4. OPTIMIZATION CRITERIA → Number of gates
5. SOLUTION

Formulating CSPs: Knapsack



$S = \{s_1, s_2, \ldots, s_n\}$

$W = \{w_1, w_2, \ldots, w_n\}$

$V = \{v_1, v_2, \ldots, v_n\}$

C = capacity

1. VARIABLES
2. DOMAINS
3. SATISFACTION CONSTRAINTS
4. OPTIMIZATION CRITERIA
5. SOLUTION

Variables: $s_1, s_2, \ldots, s_n$

Domain := $\underline{\{0, 1\}}$ ✓

$$\sum_{i=1}^{n} (s_i \cdot w_i) \le C \quad ✓$$

$$\underline{\underline{\max}} \left(\sum_{i=1}^{n} s_i \cdot v_i \right)$$

Formulating CSPs: Time Table

1. VARIABLES
2. DOMAINS
3. SATISFACTION CONSTRAINTS
4. OPTIMIZATION CRITERIA
5. SOLUTION

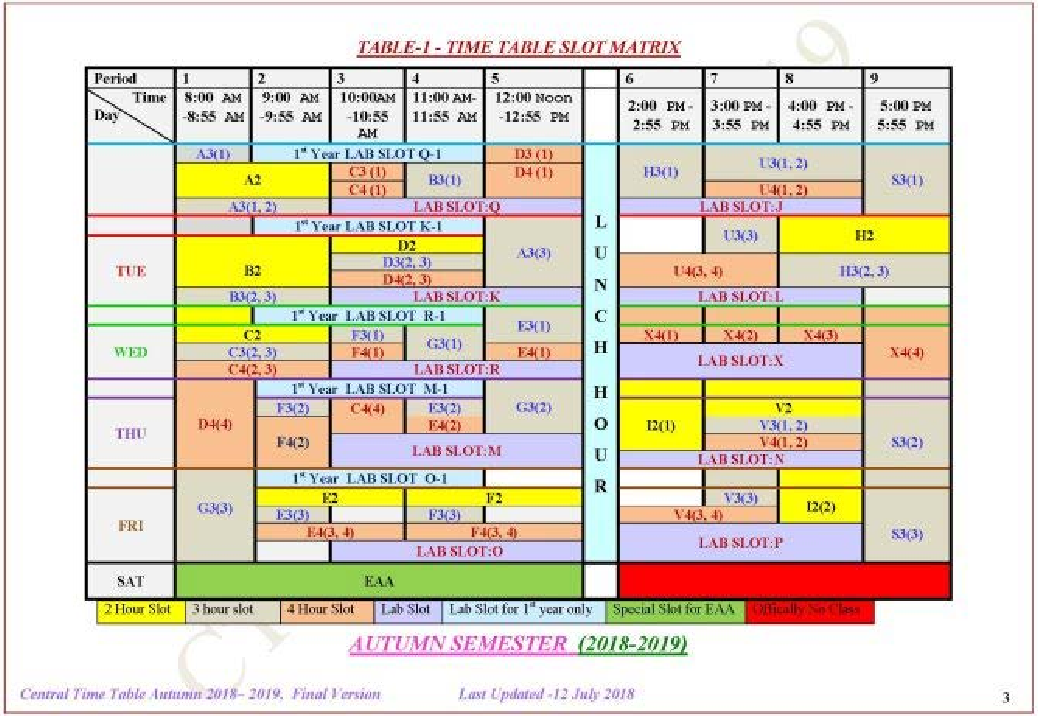


TABLE-1 - TIME TABLE SLOT MATRIX

Period / Time / Day	1 8:00 AM -8:55 AM	2 9:00 AM -9:55 AM	3 10:00AM -10:55 AM	4 11:00 AM- 11:55 AM	5 12:00 Noon -12:55 PM		6 2:00 PM - 2:55 PM	7 3:00 PM - 3:55 PM	8 4:00 PM - 4:55 PM	9 5:00 PM 5:55 PM
	A3(1); A2; A3(1, 2)	1st Year LAB SLOT Q-1	C3 (1); C4 (1); LAB SLOT:Q	B3(1)	D3 (1); D4 (1)	LUNCH HOUR	H3(1); LAB SLOT:J	U3(1, 2); U4(1, 2)		S3(1)
TUE	B2; B3(2, 3)	1st Year LAB SLOT K-1	D2; D3(2, 3); D4(2, 3); LAB SLOT:K		A3(3)		U4(3, 4); LAB SLOT:L	U3(3)	H2; H3(2, 3)	
WED	C2; C3(2, 3); C4(2, 3)	1st Year LAB SLOT R-1	F3(1); F4(1); LAB SLOT:R	G3(1)	E3(1); E4(1)		X4(1); LAB SLOT:X	X4(2)	X4(3)	X4(4)
THU	D4(4)	1st Year LAB SLOT M-1; F3(2); F4(2)	C4(4); LAB SLOT:M	E3(2); E4(2)	G3(2)		I2(1)	V2; V3(1, 2); V4(1, 2); LAB SLOT:N		S3(2)
FRI	G3(3)	1st Year LAB SLOT O-1; E2; E3(3); E4(3, 4)	LAB SLOT:O	F2; F3(3); F4(3, 4)			V4(3, 4); LAB SLOT:P	V3(3)	I2(2)	S3(3)
SAT	EAA									

2 Hour Slot	3 hour slot	4 Hour Slot	Lab Slot	Lab Slot for 1st year only	Special Slot for EAA	Officially No Class

AUTUMN SEMESTER (2018-2019)

Central Time Table Autumn 2018- 2019, Final Version   Last Updated -12 July 2018   3

Slots, Rooms, Subjects, Teachers, Students

Room-Slots: Subjects
Subjects: L-T-P, Teachers, Students

Multi-layered constraints

Intricate Optimization

free slots
total duration
minimize movement time

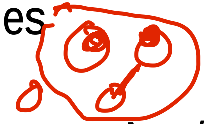
Exercise: Time-Tabling in the era of online classes

3 lectures/week

1-1-1   1-2

CSP Solution Overview

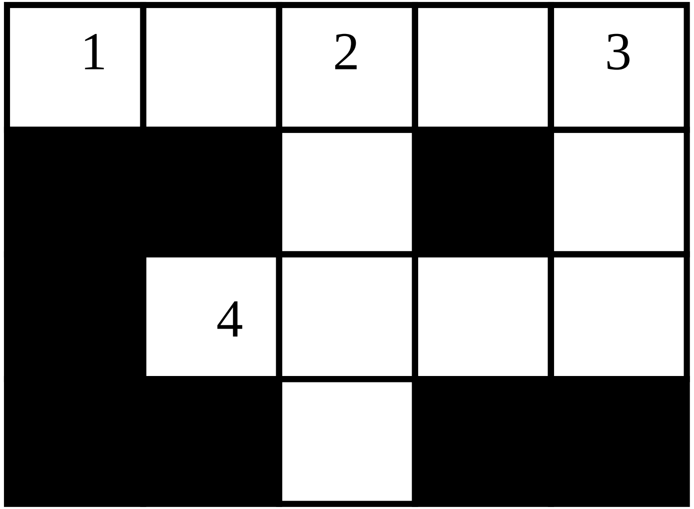
- CSP Graph Creation:
  - Create a Node for Every Variable. All possible Domain Values are initially Assigned to the Variable
  - Draw edges between Nodes if there is a Binary Constraint. Otherwise Draw a hyper-edge between nodes with constraints involving more than two variables

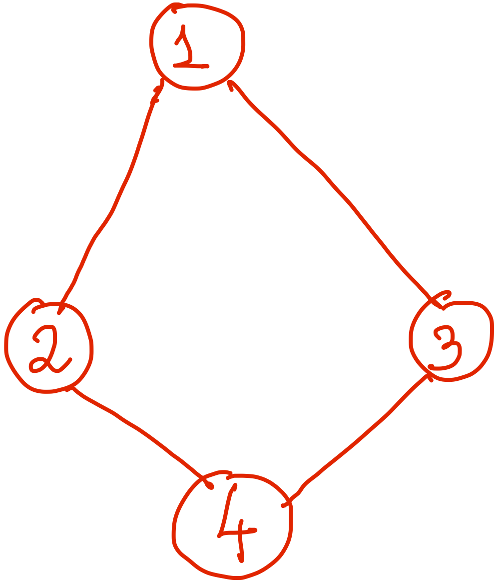


- Constraint Propagation:
  - Reduce the Valid Domains of Each Variable by Applying Node Consistency, Arc / Edge Consistency, K-Consistency, till no further reduction is possible. If a solution is found or the problem found to have no consistent solution, then terminate
- Search for Solution:
  - Apply Search Algorithms to Find Solutions
  - There are interesting properties of CSP graphs which lead of efficient algorithms in some cases: Trees, Perfect Graphs, Interval Graphs, etc
  - Issues for Search: Backtracking Scheme, Ordering of Children, Forward Checking (Look-Ahead) using Dynamic Constraint Propagation
  - Solving by Converting to Satisfiability (SAT) problems

logic problem : satisfiability

CSP Graph for Crossword



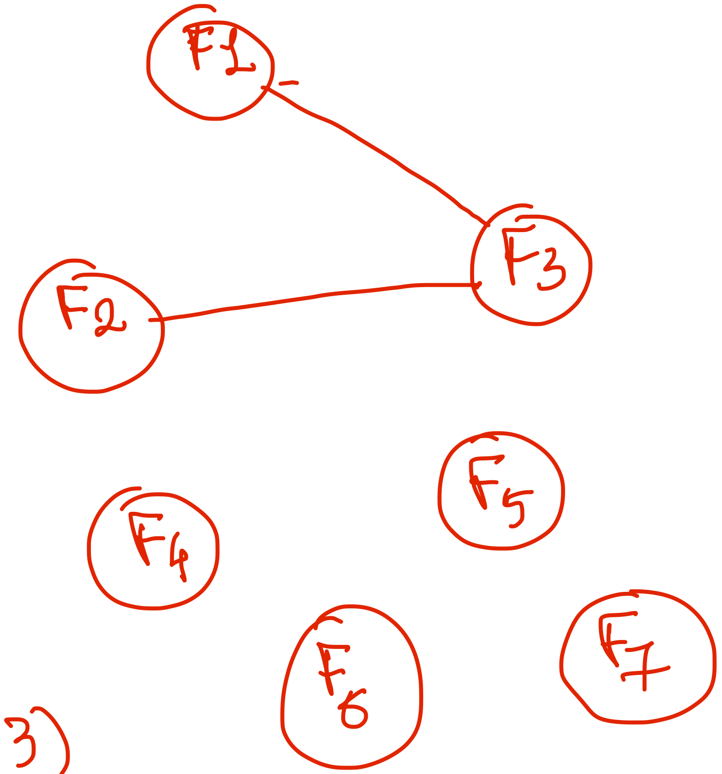


Word List:

astar, happy, hello, hoses, live, load, loom, peal, peel, save, talk, ant, oak, old

CSP Graph for Airline Gate Scheduling

Flight No	Dep Time	G Start	G End
F1	7:00	6:15	7:15
F2	8:30	7:45	8:45
F3	7:45	7:00	8:00
F4	9:45	9:00	10:00
F5	10:00	9:15	10:15
F6	9:00	8:15	9:15
F7	11:00	10:15	11:15



3 Gates (1, 2, 3)

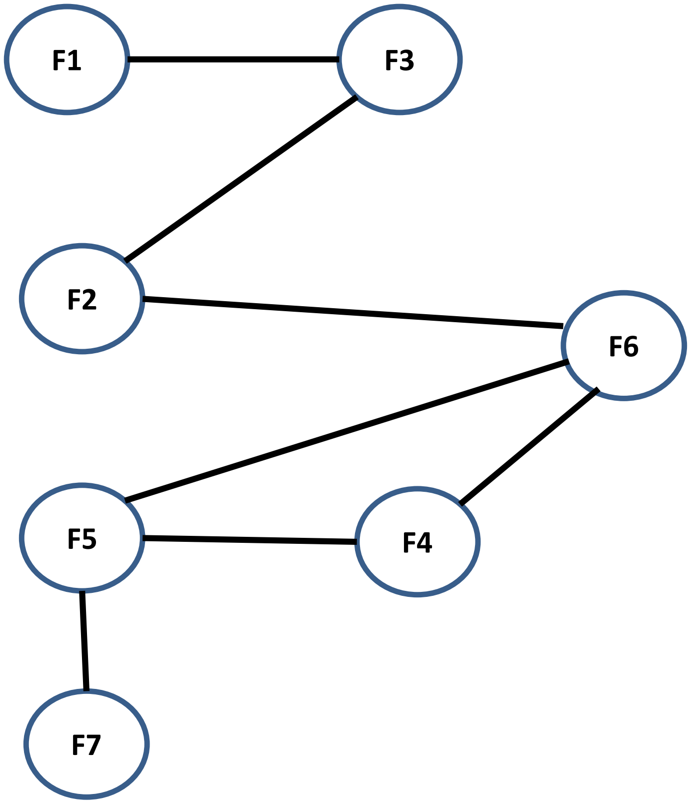
CSP Graph for Airline Gate Scheduling

Flight No	Dep Time	G Start	G End
F1	7:00	6:15	7:15
F2	8:30	7:45	8:45
F3	7:45	7:00	8:00
F4	9:45	9:00	10:00
F5	10:00	9:15	10:15
F6	9:00	8:15	9:15
F7	11:00	10:15	11:15

{1, 2, 3}

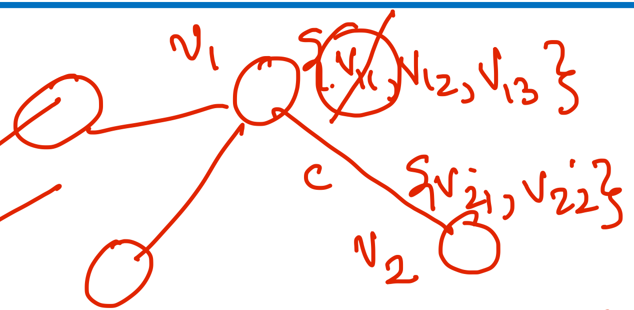
minimize

Dom: 7 Gates

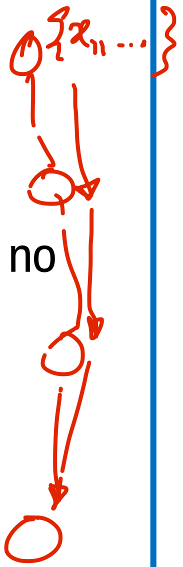


Constraint Propagation Steps

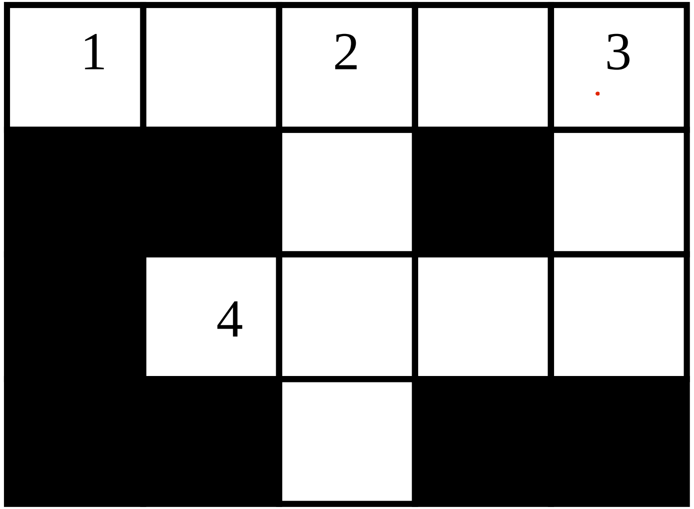
- <u>Constraints</u>
  - Unary Constraints or Node Constraints
  - Binary Constraints or Edges between CSP Nodes
  - Higher order or Hyper-Edges between CSP Nodes

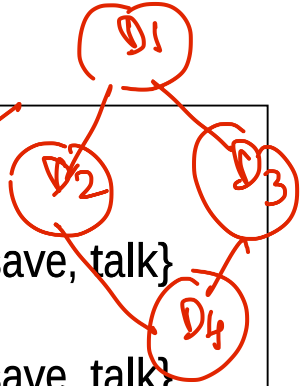


- <u>Node Consistency</u>
  - For every Variable V_i, remove all elements of D_i that do not satisfy the Unary Constraints for the Variable
  - First Step is to reduce the domains using Node Consistency
- <u>Arc Consistency</u> Edge Consistency
  - For every element x_ij of D_i, for every edge from V_i to V_j, remove x_ij if it has no consistent value(s) in other domains satisfying the Constraints
  - Continue to iterate using Arc Consistency till no further reduction happens.
- <u>K-Consistency or Path Consistency</u>
  - For every element y_ij of D_i, choose a Path of length L with L variables, use a consistency checking method similar to above to reduce domains if possible



CSP Graph for Crossword





Applying Node Consistency:

D1 = {astar, happy, hello, hoses}
D2 = {live, load, loom, peal, peel, save, talk}
D3 = {ant, oak, old}
D4 = {live, load, loom, peal, peel, save, talk}

NOW APPLY **ARC CONSISTENCY**

Applying Arc Consistency:

D1 = {~~astar~~, ~~happy~~, hello, ~~hoses~~}

D2 = {~~live~~, load, loom, peal, ~~peel~~, ~~save~~, ~~talk~~}

D3 = {~~ant~~, oak, old}

D4 = {~~live~~, load, ~~loom~~, ~~peal~~, ~~peel~~, ~~save~~, talk}

Word List:

astar, happy, hello, hoses, live, load, loom, peal, peel, save, talk, ant, oak, old

Arc Consistency Algorithm AC-3

AC-3(*csp*) // inputs - CSP with variables, domains, constraints

queue $\leftarrow$ local variable initialized to all arcs in csp
while *queue* is not empty **do**
  $(X_i, X_j) \leftarrow$ pop(queue)
  if Revise(*csp*, X_i, X_j) **then**
    if size of D_i = 0 **then return** *false*
    for each X_k **in** X_i.neighbors-$\{X_j\}$ **do**
      add (X_k, X_i) to *queue*
return *true*

$Q = \{$ all edges $\}$

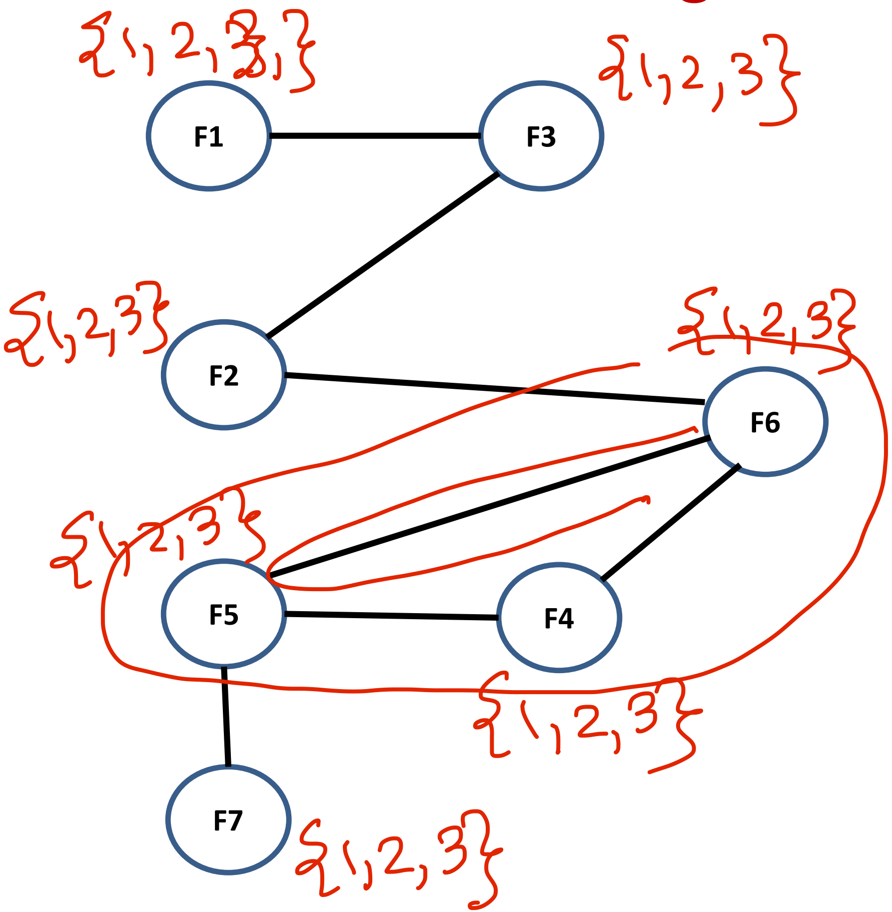
Revise(*csp*, X_i, X_j)

revised $\leftarrow$ *false*
for each *x* **in** D_i **do**
  if no value *y* in D_j allows (*x*, *y*) to satisfy constraint between X_i and X_j **then**
    delete *x* from D_i
    revised $\leftarrow$ *true*
return *revised*

Time complexity: $O(n^2d^3)$

Consistency for Airline Gate Scheduling

Flight No	Dep Time	G Start	G End
F1	7:00	6:15	7:15
F2	8:30	7:45	8:45
F3	7:45	7:00	8:00
F4	9:45	9:00	10:00
F5	10:00	9:15	10:15
F6	9:00	8:15	9:15
F7	11:00	10:15	11:15



{1, 2}

Arc

Backtracking Algorithm for CSP

Search

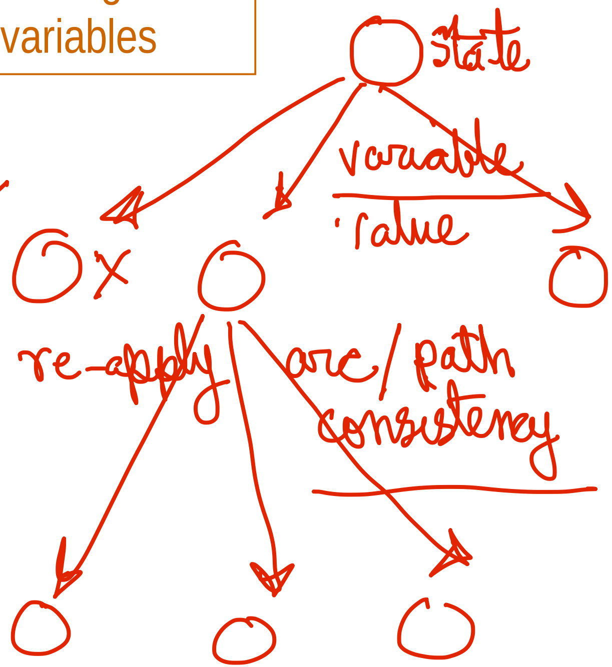
CSP-BACKTRACKING({})

DFS

CSP-BACKTRACKING(a)

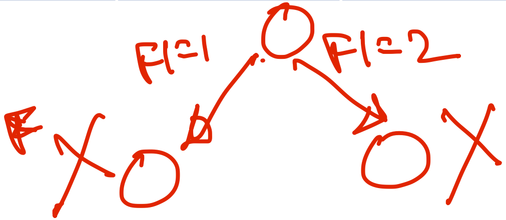
partial assignment of variables

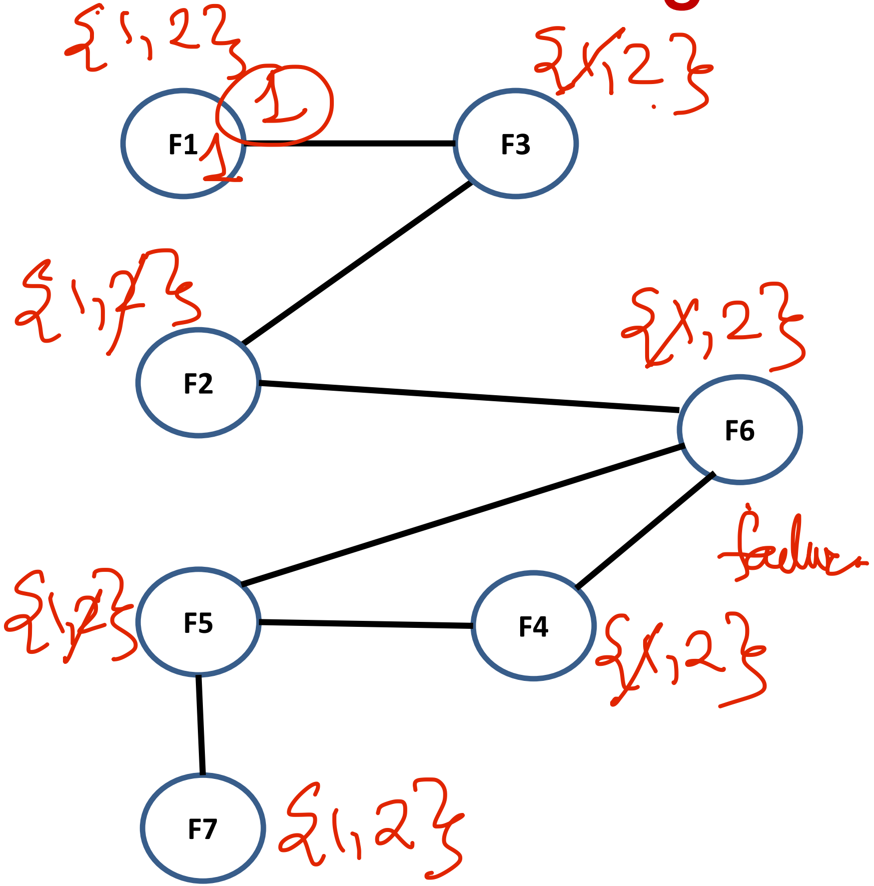
- If a is complete then return a
- X ← select unassigned variable
- D ← select an ordering for the domain of X
- For each value v in D do
  - If v is consistent with a then
    - Add (X= v) to a
    - result ← CSP-BACKTRACKING(a)
    - If result ≠ *failure* then return result
- Return *failure*



Backtracking for Airline Gate Scheduling

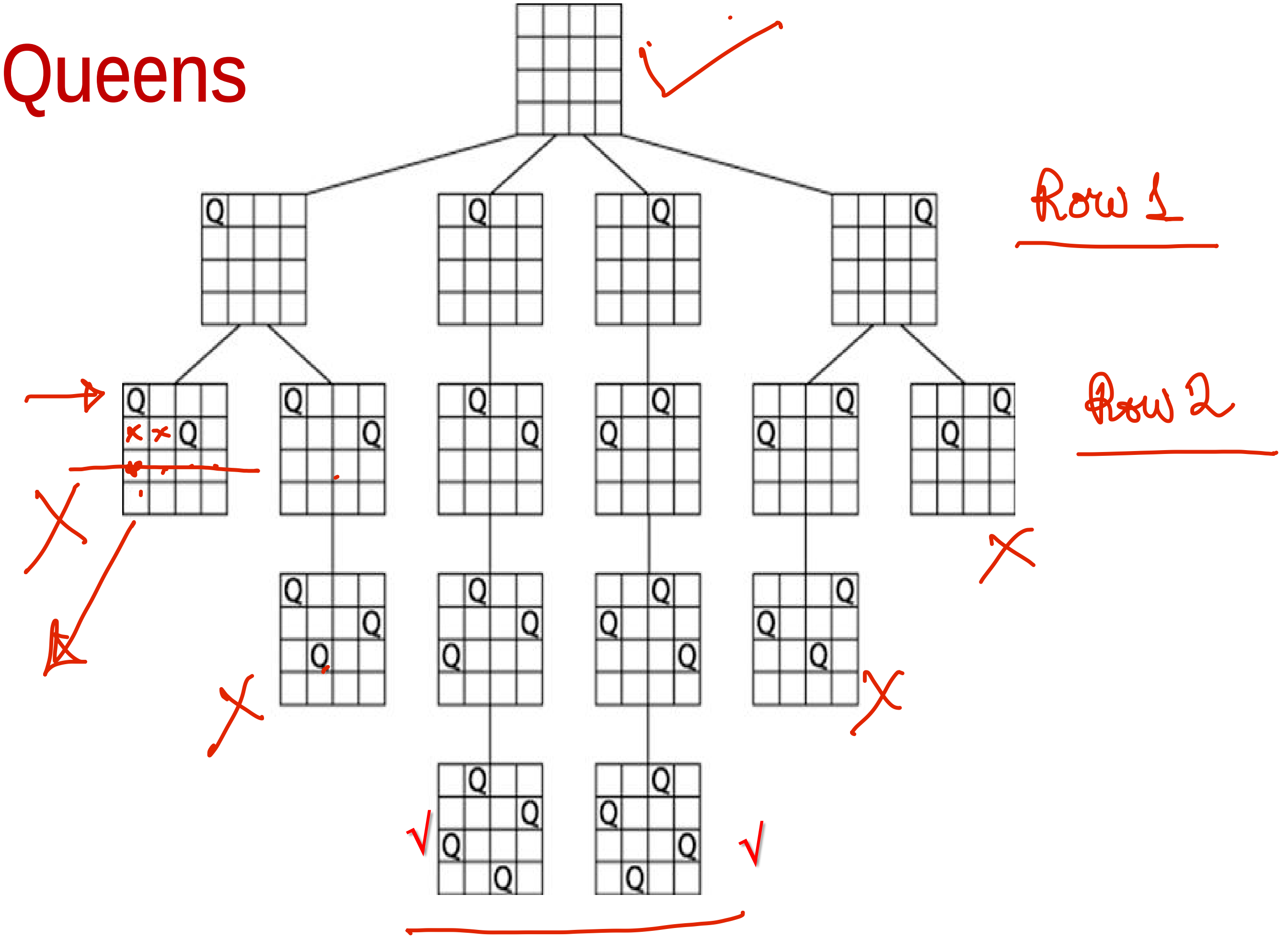
Flight No	Dep Time	G Start	G End
F1	7:00	6:15	7:15
F2	8:30	7:45	8:45
F3	7:45	7:00	8:00
F4	9:45	9:00	10:00
F5	10:00	9:15	10:15
F6	9:00	8:15	9:15
F7	11:00	10:15	11:15





Search 4-Queens

DFS

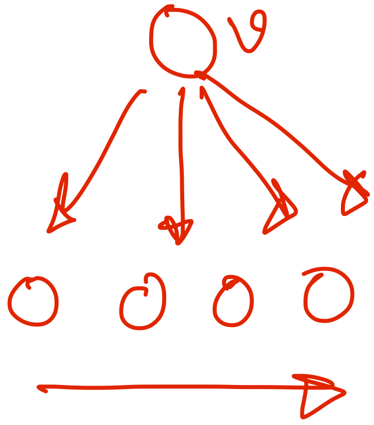


Strategies for CSP Search Algorithms

- Initial Constraint Propagation
- Backtracking Search
  - Variable Ordering
    - Most Constrained Variable / Minimum Remaining Values
    - Most Constraining Variable
  - Value Ordering
    - Least Constraining Value leaving maximum flexibility
  - Dynamic Constraint Propagation Through Forward Checking
    - Preventing useless Search ahead
  - Dependency Directed Backtracking
- SAT Formulations and Solvers
- Optimization

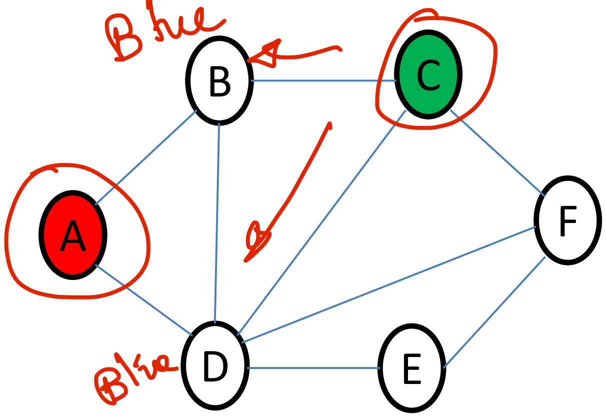
DFBB, A*, IDA*

  - Branch-and-Bound
  - SMT Solvers, Constraint Programming
- Learning, Memoizing, etc
- CSP Problems are NP-Hard in General



Forward Checking: 3 Colouring Problem

- Forward checking propagates information from assigned to unassigned variables



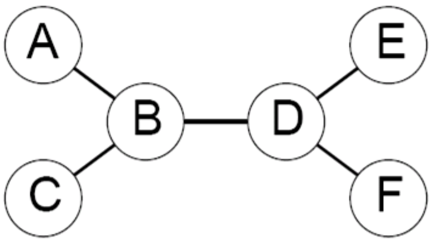
A	B	C	D	E	F
{R, G, B}	{R, G, B}	{R, G, B}	{R, G, B}	{R, G, B}	{R, G, B}
	{G, B}	{R, G, B}	{G, B}	{R, G, B}	{R, G, B}
	{B}		{B}	{R, G, B}	{R, B}

X

- B and D cannot both be blue!
- Why did we not detect this?
- Forward checking detects some inconsistencies, not all
- *Constraint propagation:* reason from constraint to constraint

Special Cases

Tree-structured CSPs



Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

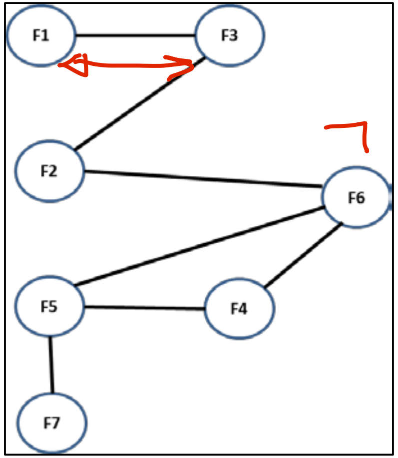
Compare to general CSPs, where worst-case time is $O(d^n)$

nd^2

For PERFECT GRAPHS, CHORDAL GRAPHS, INTERVAL GRAPHS, the Graph Colouring Problem can be solved in Polynomial Time

Solving CSP using SAT / SMT Solvers

- Boolean Satisfiability (SAT) is a CSP
- CSPs can be modelled as SAT problems
  - Try: Map Colour, Gate Scheduling, n-Queens
  - Home Exercise: Write a Generic Scheme to Convert and CSP Problem to a SAT Problem
- SAT has very efficient solvers
  - MiniSAT, CHAFF, GRASP, etc
- For Optimization cases, we can formulate them as
  - Satisfiability Modulo Theories (SMT) – with arithmetic and first order logic
  - 0/1 or Integer Linear Programming (ILP)
  - Constraint Programming Problems
  - SMT Solvers: Z3, Yices, Barcelogic, MathSAT, OpenSMT, etc



Satisfying solution

$\neg(f_{11} \wedge f_{21}) \wedge$

$\neg(f_{12} \wedge f_{22}) \wedge$

$\neg(f_{13} \wedge f_{23})$

Boolean Variables

F1: f_{11} f_{12} f_{13}

F2: f_{21} f_{22} f_{23}

$(f_{11} \vee f_{12} \vee f_{13}) \wedge (\quad)$

Thank you